NEW APPLICATIONS OF THE ABEL-LIOUVILLE FORMULA

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Abstract. In a recent paper, [1], 2005, the indefinite integrals of a certain type are calculated using some linear homogeneous differential equations of second order with variable coefficients, associated with the integrals. In some simple cases, like the examples considered in this paper, the linear independent solutions of these differential equations are directly calculated relative to elementary functions. In more difficult cases, the power series method must be used. In such situations, it is advisable to use the algebraic symbolic calculus on computer. Examples of this type will be given in a subsequent paper.

Because the main formula from which the integrals can be calculated is not rigorous proved in [1], we give here a correct proof based on the Abel-Liouville formula for the differential equations of second order. For completeness, we give here the proof for this formula and some of its applications, necessary to our work. Also, we included two examples.

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1. Abel-Liouville formula for second order linear differential equations.

We consider here the second order linear homogeneous differential equation

$$a_0(x)y''(x)+a_1(x)y'(x)+a_2(x)y(x)=0. \qquad (1)$$

with continuous coefficients $a_0(x)\neq 0$, $a_1(x)$ and $a_2(x)$.

Let $f(x)\neq 0$ be a known solution of the equation (1). Then we also have

$$a_0(x)f''(x)+a_1(x)f'(x)+a_2(x)f(x)=0. \qquad (2)$$

The determinant

$$W(y(x),f(x))=\begin{vmatrix} f(x) & y(x) \\ f'(x) & y'(x)\end{vmatrix}=f(x)y'(x)-f'(x)y(x) \qquad (3)$$

it is called the *Wronski's determinant* of the solutions $y(x)$ and $f(x)$ of the equation (1).

Theorem 1. *The Wronski determinant of the solutions* $y(x)$ *and* $f(x)$ *of the equation (1), satisfies the Abel-Liouville formula*

$$W(y(x),f(x))=ke^{u(x)}, \qquad (4)$$

where k is a constant, and

$$u(x) = -\int \frac{a_1(x)}{a_0(x)} dx . \qquad (5)$$

Proof. Eliminating the last terms of equations (1) and (2), we can write as

$$a_0(x)[f(x)y''(x) - f''(x)y(x)] + a_1(x)[f(x)y'(x) - f'(x)y(x)] = 0,$$

from which results

$$\frac{[f(x)y'(x) - f'(x)y(x)]'}{f(x)y'(x) - f'(x)y(x)} = \frac{f(x)y''(x) - f''(x)y(x)}{f(x)y'(x) - f'(x)y(x)} = -\frac{a_1(x)}{a_0(x)} .$$

By integration, it results

$$\ln|f(x)y'(x) - f'(x)y(x)| = u(x) + \ln|k| ,$$

where k is a constant. From this relation, it results Abel-Liouville formula (4).

2. Linear independence of the solutions of a linear second order differential equation

Wronski's determinants are used to determine linear independence of the solutions of a linear differential equation.

Theorem 2. *Two solutions $y(x)$ and $f(x)$ of the equation (1) are linear dependent if and only if*

$$W(y(x), f(x)) = 0 . \qquad (6)$$

Proof. If

$$W(y(x), f(x)) = f(x)y'(x) - f'(x)y(x) = 0,$$

we have

$$\frac{y'(x)}{y(x)} = \frac{f'(x)}{f(x)} .$$

By integration, from the last relation it results

$$\ln|y(x)| = \ln|Cf(x)|,$$

where C is an arbitrary constant, Therefore, we obtain $y(x)=Cf(x)$, hence the solutions $y(x)$ and $f(x)$ are linear dependent. Reciprocally, if the solutions $y(x)$ and $f(x)$ are linear dependent, we have $y(x)=Cf(x)$, hence

$$W(y(x), f(x)) = f(x)y'(x) - f'(x)y(x) = Cf(x)f'(x) - Cf(x)f'(x) = 0 .$$

Consequence 1. *Two solutions* $y(x)$ *and* $f(x)$ *of the equation (1) are linear independent if and only if*

$$W(y(x), f(x)) \neq 0 . \qquad (7)$$

The above proved Abel-Liouville formula, allows strengthening the result given in Consequence 1. Namely,

Consequence 2. *Two solutions* $y(x)$ *and* $f(x)$ *of the equation (1) are linear independent if and only if there is a value* x_0 *for which*

$$W(y(x_0), f(x_0)) \neq 0 . \qquad (8)$$

3. Solutions of second kind of the linear homogeneous differential equations of second order

If $f(x)\neq 0$ is a known solution of the equation (1), the Abel-Liouville formula can be considered as a non-homogeneous linear differential equation of the first order, from which can be obtained new solutions $g(x)$ of the equation (1), linearly independent with $f(x)$, named *solutions of second kind* of the equation (1). More exactly, we have

Theorem 3. *If* $f(x)\neq 0$ *is a known solution of the equation (1), and its coefficients satisfy the condition (5), then a second solution* $g(x)$ *of the equation, linearly independent with* $f(x)$*, is given for every constant* $k \neq 0$*, by the equation of first order,*

$$f(x)g'(x) - f'(x)g(x) = ke^{u(x)}, \qquad (9)$$

and has the form

$$g(x) = kf(x)\int \frac{e^{u(x)}}{f^2(x)}dx . \qquad (10)$$

Proof. Equation (9) results from definition (3) of the Wronski's determinant and from the Abel-Liouville formula (4). Putting equation (9) in the normal form,

$$y'(x) - \frac{f'(x)}{f(x)}y(x) = \frac{k}{f(x)}e^{u(x)},$$

and using the well-known formula for the solutions of a non-homogeneous linear differential equation of first order, with variable coefficients, we obtain the general solution of the equation (9),

$$y(x) = e^{\int \frac{f'(x)}{f(x)}dx} \left[\int e^{-\int \frac{f'(x)}{f(x)}dx} \frac{k}{f(x)} e^{u(x)} dx + C \right] = f(x) \left[k \int \frac{e^{u(x)}}{f^2(x)} dx + C \right].$$

For $C = 0$ we obtain the second type solution $y(x) = g(x)$ given by formula (10).

Direct proof. We seek the solution of the equation (1) in the form

$$y(x) = f(x)v(x), \qquad (11)$$

where $v(x)$ is the new unknown function of the equation (1). Because $y'(x) = f'(x)v(x) + f(x)v'(x)$ and $y''(x) = f''(x)v(x) + 2f'(x)v'(x) + f(x)v''(x)$, the equation (1) receives the form

$$a_0(x)f(x)v''(x) + [2a_0(x)f'(x) + a_1(x)f(x)]v'(x) +$$

$$+ [a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v(x) = 0. \qquad (12)$$

Using the relation (2), the equation (12) becomes

$$a_0(x)f(x)v''(x) + [2a_0(x)f'(x) + a_1(x)f(x)]v'(x) = 0. \qquad (13)$$

Considering the new unknown function

$$w(x) = v'(x), \qquad (14)$$

the equation (13) receives the form

$$a_0(x)f(x)w'(x) + [2a_0(x)f'(x) + a_1(x)f(x)]w(x) = 0. \qquad (15)$$

Dividing the equation (15) with $a_0(x)f(x)w(x)$ and using the relation (3), the equation (15) becomes

$$\frac{w'(x)}{w(x)} + 2\frac{f'(x)}{f(x)} - u'(x) = 0. \qquad (16)$$

The equation (16) has the solution

$$w(x) = k\frac{e^{u(x)}}{f^2(x)} \qquad (17)$$

From (14) and (17) it results

$$v(x) = \int w(x)dx + C = k\int \frac{e^{u(x)}}{f^2(x)}dx + C, \qquad (18)$$

Finally, from (11) and (18), it results for $C = 0$ and $y(x) = g(x)$, formula (10).

4. A new form for homogeneous linear differential equations of second order

Theorem 5. *The equation (1) has the solution $f(x)$ and its coefficients satisfy the condition (5), where $u(x)$ is a given differentiable function, if and only if the equation has the form*

$$f(x)y''(x) - f(x)u'(x)y'(x) + [f'(x)u'(x) - f''(x)]y(x) = 0. \quad (20)$$

In this case the second kind solution of the equation is given by the formula (10).

Proof. From (5) it results $a_1(x) = -u'(x)a_0(x)$. If $f(x)$ is a solution of the equation (1), we have (2), hence $a_2(x) = \dfrac{f'(x)u'(x) - f''(x)}{f(x)}$. Substituting these values of coefficients in equation (1), this acquires form (20).

Conversely, it is obviously that equation (20) has $f(x)$ as solution and its coefficients satisfy the condition (5).

The last statement of theorem 5, it follows from theorem 3.

5. Computing indefinite integrals by linear homogeneous differential equations of second order

If we know two linear independent solutions of the equation (1), then from the above results, which are derived from the Abel-Liouville formula, we can calculate the indefinite integrals of the form

$$I = \int \frac{e^{u(x)}}{f^2(x)}dx. \quad (21)$$

Namely, we have the following main result

Theorem 6. *If $u(x)$ and $f(x)$ are given functions, first once and second twice times differentiable, then the indefinite integral (21) can be calculated from the formula*

$$I = \int \frac{e^{u(x)}}{f^2(x)}dx = \frac{g(x)e^{u(x)}}{f(x)[f(x)g'(x) - f'(x)g(x)]} + C, \quad (22)$$

where $g(x)$ is a second kind solution of the linear differential equation (20) and C is an arbitrary constant.

Proof. The formula (22) results from the above formulas (9) and (10) from theorem 3. In conformity with theorem 5, the functions $f(x)$ and $g(x)$ are the linear independent solutions of the differential equation (20).

Remark. To calculate indefinite integrals of type (21) by this new method, must be done the following steps:

1) The functions $u(x)$ and $f(x)$ must be identified from the integral (21);

2) Using the functions $u(x)$ and $f(x)$, the homogeneous linear differential equation (20) for which the function $f(x)$ is solution; is determined.
3) The solution of second kind $g(x)$ of the differential equation (20) is obtained by direct methods or using power series. The formula (10) can not be used to calculation of $g(x)$, because it contains the integral to be calculated;
4) Using the functions $u(x)$, $f(x)$ and $g(x)$, the indefinite integral (21) is calculated from the formula (22).

6. Examples

We give two examples that was also considered in [1]. In that article for second example has been used the power series method, which is not necessary.

1. To calculate the indefinite integral $I = \int e^x dx$, we rewrite the integral as $I = \int \frac{e^{3x}}{(e^x)^2} dx$. We choose $u(x) = 3x$ and $f(x) = e^x$. Substituting these functions in (20) we obtain the second order linear homogeneous differential equation

$$y''(x) - 3y'(x) + 2y(x) = 0,$$

with the two linearly independent solutions $f(x) = e^x$ and $g(x) = e^{2x}$. Using the formula (22), it results that the integral is

$$I = \int e^x dx = \frac{e^{2x}e^{3x}}{e^x\left(2e^{2x}e^x - e^{2x}e^x\right)} + C = e^x + C.$$

2. To calculate the integral $\int \frac{xdx}{\sin^2(x^2)}$, $x > 0$, $x \neq \sqrt{n\pi}$, $n = 1,2,\ldots$, we choose $u(x) = \ln x$ and $f(x) = \sin(x^2)$. In this case, the equation (20) takes the form

$$xy''(x) - y'(x) + 4x^3 y(x) = 0.$$

Making the change of variables $t = x^2$, we have $y'(x) = y'(t)\frac{dt}{dx} = 2xy'(t)$, and $y''(x) = 2y'(t) + 2xy''(t)\frac{dt}{dx} = 2y'(t) + 4ty''(t)$, hence the differential equation becomes

$$y''(t) + y(t) = 0,$$

with linear independent solutions $\cos t$ and $\sin t$ .Therefore $g(x) = \cos(x^2)$, and the integral is given by the formula

$$\int \frac{xdx}{\sin(x^2)^2} = \frac{x\cos\left(x^2\right)}{-2x\sin(x^2)\left(\sin^2(x^2)+\cos^2\left(x^2\right)\right)} + C = -\frac{1}{2}ctg(x^2) + C.$$

Remark 1. Using the change of variables $t = x^2$, we have

$$\int \frac{xdx}{\sin^2\left(x^2\right)} = \frac{1}{2}\int \frac{dt}{\sin^2 t} = -\frac{1}{2}ctg\left(t\right) + C = -\frac{1}{2}ctg\left(x^2\right) + C,$$

Remark 2. These examples show that by the new method of calculating the indefinite integrals, the usual results are obtained.

References:

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